Graph spanners

Laurent Viennot

June 3rd, 2010
From a graph $G$
Compute a subgraph $H$ spanning $G$
Graph spanner

Definition
A spanner $H$ of a graph $G$ is a subgraph of $G$ with:

- few edges,
- short distances.

Trade-off number of edges vs stretch of distances.
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What for?

- Implicit pre-processing step of approximate distance oracle computation [Thorup & Zwick 2005].
- By-product of compact routing schemes [Peleg & Upfal 1989].
Compact routing

**Definition**

A compact routing scheme for a graph $G$ consists in designing routing tables with:

- **small size,**
- **short routes.**

**Trade-off** table size vs stretch of routes.

**Examples:**

- **Classical routing:** one entry per destination (size $O(n)$ per node).
- **Grid:** use coordinates (size $O(1)$ per node).
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- Grid: use coordinates (size $O(1)$ per node).
BGP entries in the Internet

Internet: use prefix of addresses.
Compact routing theory

- Explicit trade-off between table size and route length [Peleg89], [Gavoille96], [Thorup01],...
- Static centralized solutions: [Thorup01], [Brady06], [Abraham04], [Abraham08],...
- Challenge: dynamic distributed compact routing.
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From compact routing to spanner construction

- Take a compact routing scheme for $G$.
- For each node add to $H$ the link to each neighbor listed in its routing table.
- $H$ is a spanner of $G$.
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Graph spanner

Definition (Peleg & al. 1987-89)

Given an undirected graph $G$, a subgraph $H \subseteq G$ is an $(\alpha, \beta)$-spanner of $G$ iff for all $u, v$,

$$d_H(u, v) \leq \alpha \cdot d_G(u, v) + \beta$$

- $\alpha$ : multiplicative stretch
- $\beta$ : additive stretch
- $m(H)$ : size (number of edges)
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Example

G

H

T
A Greedy Algorithm [Althöfer & al. 1993]

\[ H := \emptyset \]
For each edge \( uv \in E(G) \) do
  \[ \text{If } d_H(u, v) > 2k - 1 \text{ then add } uv \text{ to } H. \]

- \( H \) is a \((2k - 1, 0)\)-spanner of \( G \).
- \( H \) has girth \( g(H) > 2k \) implying \( m(H) < n^{1+1/k} \).
- This is optimal assuming the girth conjecture.
Theorem (Folklore, see Bollobás or Matoušek)
A graph \( H \) with girth \( g(H) > 2k \) has \( m(H) \leq n + n^{1+1/k} \) edges.

- **Case 2** \( 2k \): \( H \) contains an induced subgraph with minimal degree \( \delta \geq \frac{1}{2} \bar{d} \). The graph induced by nodes at distance \( \leq k \) from some \( u \) is a tree implying \( (\delta - 1)^k \leq n \).
The previous bound seems tight:

- For \( k = 1 \) (or girth 3, 4): consider \( K_{n/2,n/2} \).
- For \( k = 2 \) (or girth 5, 6): consider the finite projective plane of order \( \approx n^{1/2} \) and the bipartite graph of point-line incidences.

**Conjecture (Erdős 1964, see Erdős & Simonovits 1982)**

For any \( k \geq 1 \), there exist graphs with \( \Omega(n^{1+1/k}) \) edges and girth greater than \( 2k \).

- Proved for \( k = 1, 2, 3, 5 \) [see Wenger 1991].
- There exist graphs with \( \Omega(n^{1+2/3k}) \) edges and girth greater than \( 2k \) [Lazebnik & al. 1995].
Erdős-Simonovits Girth Conjecture

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Beyond the Girth Conjecture

- With stretch $(2k - 1, 0)$, can we get $O(f(n, m, k))$ edges?
- E.g., $O(n d^{1/k})$?
Distributed Spanner Computation
The problem

- $G$ is the communication network of a distributed system.
- Compute an $(\alpha, \beta)$-spanner of $G$ in the LOCAL model: synchronous rounds, unbounded message size.
Lower Bound

Theorem (Elkin 2006, DGPV’08)

Assuming the girth conjecture, an algorithm that computes a connected subgraph with $o(n^{1+1/k})$ edges has expected time at least $k$.

- A graph with girth greater than $2k$ looks like a tree after $t \leq k$ rounds.
Upper Bound

Theorem (DGPV'08)

It is possible to compute a $(2k - 1, 0)$-spanner with $O(kn^{1+1/k})$ edges in $k$ rounds (in $3k - 1$ rounds if $n$ is unknown).
Per Node Cluster Growth [DGPV'08]
Set $\sigma(u)$ to any value in $[\max_{v \in B(u, k-1)} |B(v, k)|^{1/k}, n^{1/k}]$.

$C(u) := \{u\}$ /* cluster around $u$ */

$H(u) := (\{u\}, \emptyset)$ /* spanner edges selected by $u$ */

For $i := 1$ to $k$ do

Node $u$ sends $C(u)$ to all nodes in $N(u)$, and receives $C(v)$ from all $v \in N(u)$.

$W(u) := N(u) \setminus \{v \mid C(u) \cap C(v) \neq \emptyset\}$ /* nodes to cover */

$j := 0$

While $\exists w \in W(u)$ and $j < \sigma(u)$ do

Pick $w \in W(u)$.

Add edge $uw$ to $H(u)$.

Add $C(w)$ to $C(u)$.

$W(u) := W(u) \setminus \{v \in W(u) \mid C(v) \cap C(w) \neq \emptyset\}$

$j := j + 1$
Without Knowing $n$ (Performances)

- **Time**: $3k - 1$ rounds ($k$ if $n$ is known).
- **Edges**: $m(H) \leq \sum_u k\sigma(u) \leq kn\Delta_k^{1/k} \leq kn^{1+1/k}$ where $\Delta_k = \max_u |B(u, k)|$.
- After iteration $i$, radius($C(u)$) $\leq i$.
- After iteration $i$, $|C(u)| \geq \max_{v \in B(u, k-i)} |B(v, k)|^{i/k}$ or $W(u) = \emptyset$.
- **Stretch**: at the end, $W(u) = \emptyset$ and for $v \in N(u)$, $d_H(u, v) \leq 1 + (k - 1) + (k - 1)$. 

⇐ $\Rightarrow$ $\frac{1}{2}$ $\frac{2}{3}$
Without Knowing $n$ (Performances)

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Related Upper Bounds

- [Baswana et al. 2007] provide an algorithm for computing a \((k, k - 1)\)-spanner (unweighted) with \(O(kn^{1+1/k})\) expected size in \(O(k)\) rounds using randomized sampling:
  - at round \(i\), a cluster is considered for growing with probability \(n^{-1/k}\).
  - It is randomized and requires knowledge of \(n\).
- [Baswana et al. 2007] provide an algorithm for computing a \((2k - 1, 0)\)-spanner (weighted) with \(O(kn^{1+1/k})\) expected size in \(O(k^2)\) rounds using also randomized sampling and knowledge of \(n\).
Toward Additive Spanners
The Open Problem

- Do $(1, 2k - 2)$-spanners with $O(n^{1+1/k})$ edges exist?
  - $k = 2$: yes [Aingworth & al. 1999].
  - $k = 3$: ok with stretch $(1, 6)$ [Baswana & al. 2005].
  - $k > 3$: ??
- Do $(1, O(1))$-spanners with $o(n^{4/3})$ edges exist?
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(1, 2)-spanners of size $O(n^{3/2})$ [Aingworth & al. 1999]

\[
G' := G \\
H := \emptyset \\
\textbf{While } \exists u \in V(G') \mid \text{deg}_{G'}(u) > n^{1/2} \textbf{ do} \\
\quad C(u) := B_{G'}(u, 1) \quad \text{/* cluster around } u \text{ */} \\
\quad G' := G' - C(u) \\
\quad H := H \cup \text{BFS}_{G}(u) \\
H := H \cup E(G') \quad \text{/* add remaining edges */}
\]

- $H$ is a (1, 2)-spanner of $G$.
- $m(H) \leq 2n^{3/2}$ (at most $n^{1/2}$ clusters).
Proof of stretch
(1, 6)-spanners of size $O(n^{4/3})$ [Baswana & al. 2005]

- Greedily compute clusters of size greater than $n^{1/3}$.
- At most $n^{2/3}$ clusters.
- Add shortest paths for reducing distances between all cluster pairs.
- This works because $(n^{2/3})^2 = n^{4/3}$.
- For $k = 4 : \left( n^{3/4} \right)^2 \gg n^{5/4}$...
Nearly Additive Spanners

- $(1 + \varepsilon, \beta)$-spanners with $O(\beta n^{1+1/k})$ edges with $
abla = k^{\log \log k - \log \varepsilon}$ [Elkin & Peleg 2004].
- $(1 + \varepsilon, O(1/\varepsilon)^{k-2})$-spanners with $O(kn^{1+1/k})$ edges [Thorup & Zwick 2006], indeed $f$-spanners with $f(d) = d + O \left( kd^{1 - \frac{1}{k-1}} \right)$. 
Set \( \sigma(u) \) to a value in
\[ \max_{v \in B(u, \rho[2,k])} |B(v, \rho[1,k])|^{1/k}, n^{1/k} \]
\( C(u) := \{u\} \) /* cluster around \( u \) */
\( F(u) := \text{FALSE} \) /* termination flag */
\( H(u) := (\{u\}, \emptyset) \) /* spanner edges selected by \( u \) */

For \( i := 1 \) to \( k \) do

Node \( u \) sends \( C(u) \), \( F(u) \) to all nodes in \( B(u, \rho_i) \),
and receives \( C(v), F(v) \) from all \( v \in B(u, \rho_i) \).
\( W(u) := B(u, \rho_i) \setminus \{v \mid F(v) = \text{TRUE} \text{ or } C(u) \cap C(v) \neq \emptyset\} \)
\( j := 0 \)

While \( \exists w \in W(u) \) and \( j < \sigma(u) \) do

Pick \( w \in W(u) \) such that \( d_G(u, w) \) is minimal.
Add a shortest path in \( G \) from \( u \) to \( w \) to \( H(u) \).
Add \( C(w) \) to \( C(u) \).
\( W(u) := W(u) \setminus \{v \in W(u) \mid C(v) \cap C(w) \neq \emptyset\} \)
\( j := j + 1 \)

If \( W(u) = \emptyset \) then \( F(u) := \text{TRUE} \) else \( F(u) := \text{FALSE} \)
Performances

- **Time:** $O(\rho_1 + \cdots + \rho_k)$.
- **Edges:** $m(H) \leq (\rho_1 + \cdots + \rho_k)n^{1+1/k}$.
- **After iteration $i$, radius($C(u)$) $\leq \rho_1 + \cdots + \rho_i$.**

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- **Edges**: $m(H) \leq (\rho_1 + \cdots + \rho_k)n^{1+1/k}$.
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Conclusion

• A challenge: dynamic distributed compact routing.
• A first step: distributed spanner construction.
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Spanner Variants
Spanners and link state routing

Link state routing:
- Each node discovers its neighbors,
- and advertises the state of some neighboring links.

Optimize link state advertisements:
- few links (flooded information),
- efficient routes.
Dense network $G$
Sub-Graph $H$
Augmented sub-graph $H_u$
Definition (Remote Spanner, JV’09)

\[ H \subseteq G \text{ is an } (a, \beta)\text{-remote-spanner of } G \text{ iff } \]
\[ d_{H_u}(u, v) \leq a \cdot d_G(u, v) + \beta \text{ for all } u, v \text{ where } \]
\[ H_u = H \cup \{uv \mid v \in N(u)\}. \]

OLSR relies on the construction of a \((1, 0)\)-remote-spanner.
Remote Spanners

Theorem (JV’09)

- An \((a, \beta)\)-spanner is an \((a, \beta - a + 1)\)-remote-spanner implying the existence of \((k, 0)\)-spanner with \(O(kn^{1+1/k})\) edges using [Baswana & al. 2005].

- A random unit disk graph has a \((1, 0)\)-remote-spanner with \(O(n^{4/3})\) edges in expectation.

- A \((1, 0)\)-remote-spanner with size \(O(\log n)\) from optimal can distributively be computed in \(O(1)\) time.
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Theorem (JV’09)

- If $G$ is the unit ball graph of a doubling metric with dimension $p$ (distances are unknown), a $(1 + \varepsilon, 1 - 2\varepsilon)$-remote-spanner with $O(n\varepsilon^{-(p+1)})$ edges can be computed in $O(\varepsilon^{-1})$ time.

- If $G$ is the unit ball of a doubling metric, a 2-multipath $(2, -1)$-remote-spanner ($\varepsilon = 1$) with $O(n)$ edges can be computed in $O(1)$ time.

- A $c$-multipath $(1, 0)$-remote-spanner with size $O(\log n)$ from optimal can distributively be computed in $O(1)$ time.
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- If $G$ is the unit ball of a doubling metric, a 2-multipath $(2, -1)$-remote-spanner ($\varepsilon = 1$) with $O(n)$ edges can be computed in $O(1)$ time.
- A $c$-multipath $(1, 0)$-remote-spanner with size $O(\log n)$ from optimal can distributively be computed in $O(1)$ time.
Spanner of Directed Graphs

Definition (Roundtrip Distance, Cowen & Wagner 1999)
In a strongly connected graph $G$ the roundtrip distance $d_G(u, v)$ is the weight of a lightest circuit traversing $u$ and $v$:

$$d_G(u, v) = d_G(u, v) + d_G(v, u)$$

Theorem (Roditty & al. 2002)
Every graph has a $(3, 0)$-roundtrip-spanner with $O(n^{3/2})$ edges and a $(2k + \varepsilon, 0)$-roundtrip-spanner with $O(k^2 n^{1+1/k} \log nW)$ edges where the weights are in the range $[1, W]$. 

⇐ ? ⇒
Spanner of Directed Graphs

**Definition (Roundtrip Distance, Cowen & Wagner 1999)**
In a strongly connected graph $G$ the roundtrip distance $d_G(u, v)$ is the weight of a lightest circuit traversing $u$ and $v$:

$$d_G(u, v) = \overrightarrow{d}_G(u, v) + \overrightarrow{d}_G(v, u)$$

**Theorem (Roditty & al. 2002)**
Every graph has a $(3, 0)$-roundtrip-spanner with $O(n^{3/2})$ edges and a $(2k + \varepsilon, 0)$-roundtrip-spanner with $O\left(\frac{k^2}{\varepsilon} n^{1+1/k} \log nW\right)$ edges where the weights are in the range $[1, W]$. 
Multipath Spanners

Definition (Multipath Distance, JV'09)
The $c$-multipath distance $d_G^c(u,v)$ is the weight of the lightest collection of $c$ disjoint paths from $u$ to $v$.

Definition (Multipath Spanner, JV'09)
$H \subseteq G$ is a $c$-multipath $(a, \beta)$-spanner of $G$ iff $d_H^i(u,v) \leq a \cdot d_G^i(u,v) + i\beta$ for all $u,v$ and $i \leq c$.

Theorem (GGV'10)
Every graph has a 2-multipath $(3,0)$-spanner with $O(n^{3/2})$ edges and a $c$-multipath $(c(2k - 1), 0)$-spanner with $O(cn^{1+1/k})$ edges (edge disjoint paths are considered).
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Definition (Fault Tolerant Spanner, Chechik & al. 2009)

$H \subseteq G$ is a $f$-fault tolerant $(a, \beta)$-spanner of $G$ iff

$$d_{H-F}(u, v) \leq ad_{G-F}(u, v) + \beta \text{ for all } u, v \text{ and } F \subseteq V(G) \text{ with } n(F) \leq f.$$ 

Theorem (Chechik & al. 2009)

Every graph has an $f$-fault tolerant $(2k - 1, 0)$-spanner with

$$O(f^3 k^{f+1} \cdot n^{1+1/k} \log^{1-1/k} n) \text{ edges.}$$
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Every graph has an $f$-fault tolerant $(2k - 1, 0)$-spanner with $O(f^{3} k^{f+1} \cdot n^{1+1/k} \log^{1-1/k} n)$ edges.
Definition (Distance Emulator, Dor & al. 2000)

$H$ is an $(\alpha, \beta)$-emulator of $G$ iff for all $u, v$, $d_G(u, v) \leq d_H(u, v) \leq \alpha \cdot d_G(u, v) + \beta$.

Theorem (Dor & al. 2000)

Every graph has $(1, 4)$-emulator with $O(n^{3/2})$ edges.

Theorem (Thorup & Zwick 2006)

Every graph has an $f$-emulator with $O(kn^{1+\frac{1}{2k-1}})$ edges where $f(d) = d + O\left(kd^{1-\frac{1}{k-1}}\right)$. 
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Distance Preservers

Definition (Bollobás & al. 2003)

\[ H \subseteq G \] is a \( D \)-preserver iff \( d_H(u, v) = d_G(u, v) \) for all \( u, v \) such that \( d_G(u, v) \geq D \).

Theorem (Bollobás & al. 2003)

Every graph has a \( D \)-preserver with \( O(n^2/D) \) edges (and this is optimal).
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Conclusion

There are still new spanner algorithms to find.