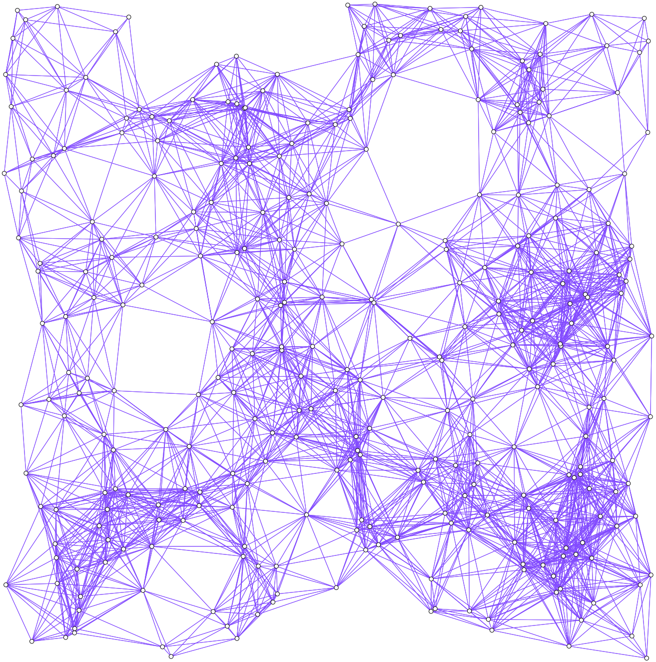
Graph spanners

Laurent Viennot

June 3rd, 2010

From a graph G



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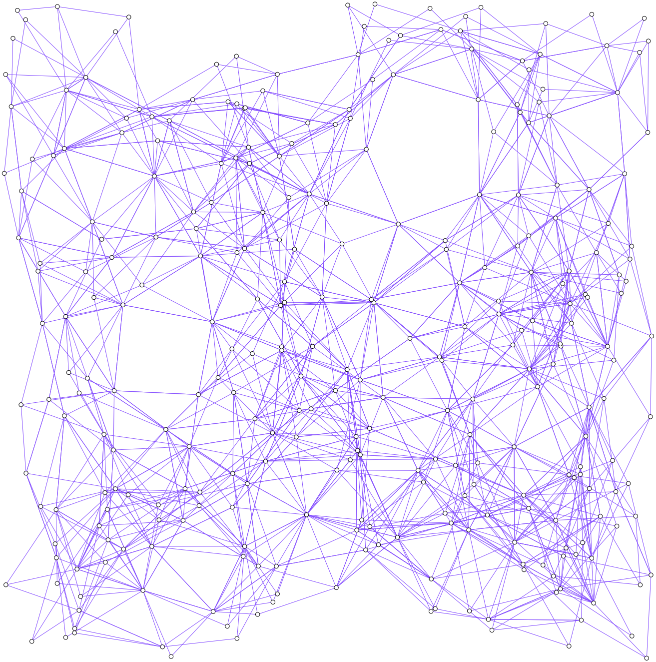
Compute a subgraph H spanning G



Graph spanner

Definition

A spanner H of a graph G is a subgraph of G with :

- *few edges,*
- *short distances.*

Trade-off number of edges vs stretch of distances.

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What for?

- Synchronizer [Awerbuch 1985, Peleg & Ullman 1989].
- Implicit pre-processing step of approximate distance oracle computation [Thorup & Zwick 2005].
- By-product of compact routing schemes [Peleg & Upfal 1989].

Compact routing

Definition

A compact routing scheme for a graph G consists in designing routing tables with :

- *small size,*
- *short routes.*

Trade-off table size vs stretch of routes.

Examples :

- Classical routing : one entry per destination (size $O(n)$ per node).
- Grid : use coordinates (size $O(1)$ per node).

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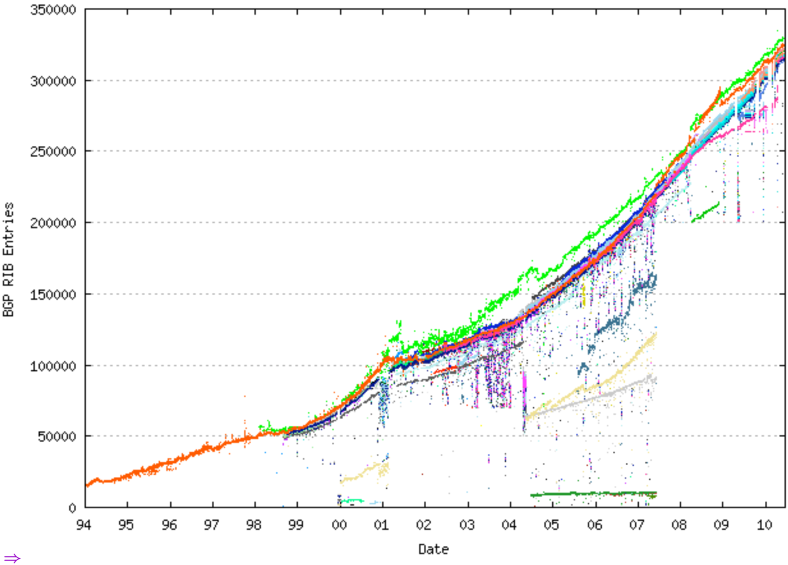
BGP entries in the Internet

Internet : use prefix of addresses.



Compact routing theory

- Explicit trade-off between table size and route length [Peleg89], [Gavoille96], [Thorup01],...
- Static centralized solutions : [Thorup01], [Brady06], [Abraham04], [Abraham08],...
- Challenge : dynamic distributed compact routing.

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From compact routing to spanner construction

- Take a compact routing scheme for G.
- For each node add to H the link to each neighbor listed in its routing table.
- H is a spanner of G.
- Challenge for today : distributed spanner construction.

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Graph spanner

Definition (Peleg & al. 1987-89)

Given an undirected graph G, a subgraph $H \subseteq G$ is an (α, β)-spanner of G iff for all u, v,

$$d_H(u, v) \leq \alpha \cdot d_G(u, v) + \beta$$

- α : multiplicative stretch
- β : additive stretch
- $m(H)$: size (number of edges)

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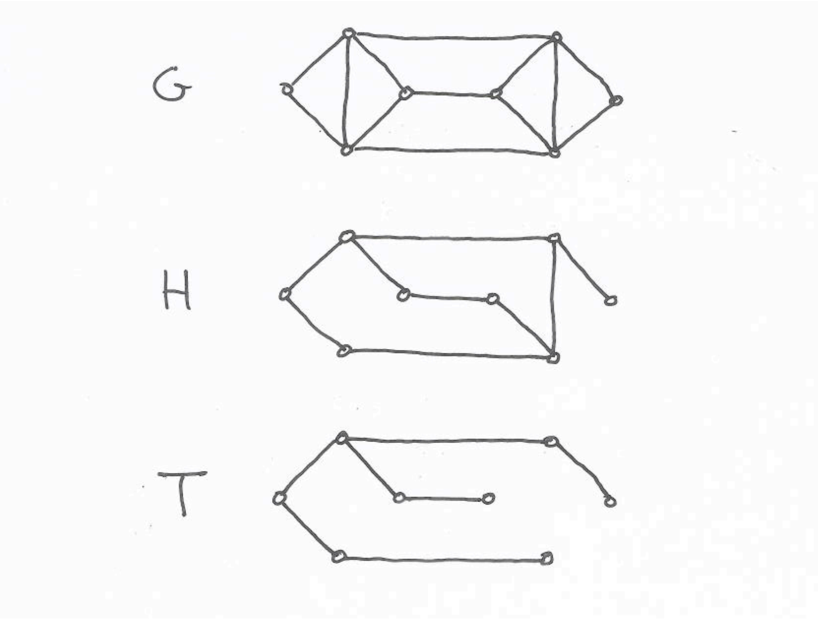
Example



A Greedy Algorithm [Althöfer & al. 1993]

$H := \emptyset$
For each *edge* $uv \in E(G)$ **do**
  If $d_H(u, v) > 2k - 1$ **then** add uv to H.

- H is a $(2k - 1, 0)$-spanner of G.
- H has girth $g(H) > 2k$ implying $m(H) < n^{1+1/k}$.
- This is optimal assuming the girth conjecture.

Large Girth, Few Edges

Theorem (Folklore, see Bollobás or Matoušek)

A graph H with girth $g(H) > 2k$ has $m(H) \le n + n^{1+1/k}$ edges.

- Case $2k$: H contains an induced subgraph with minimal degree $\delta \ge \frac{1}{2}\overline{d}$. The graph induced by nodes at distance $\le k$ from some u is a tree implying $(\delta - 1)^k \le n$.

Erdös-Simonovits Girth Conjecture

- The previous bound seems tight :
- For $k = 1$ (or girth $3, 4$) : consider $K_{n/2,n/2}$.
- For $k = 2$ (or girth $5, 6$) : consider the finite projective plane of order $\approx n^{1/2}$ and the bipartite graph of point-line incidences.

Conjecture (Erdös 1964, see Erdös & Simonovits 1982)

For any $k \geq 1$, there exist graphs with $\Omega(n^{1+1/k})$ edges and girth greater than $2k$.

- Proved for $k = 1, 2, 3, 5$ [see Wenger 1991].
- There exist graphs with $\Omega(n^{1+2/3k})$ edges and girth greater than $2k$ [Lazebnik & al. 1995].

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Beyond the Girth Conjecture

- With stretch $(2k-1, 0)$, can we get $O(f(n, m, k))$ edges?
- E.g., $O(n\overline{d}^{1/k})$?

Distributed Spanner Computation

The problem

- G is the communication network of a distributed system.
- Compute an (α, β)-spanner of G in the LOCAL model : synchronous rounds, unbounded message size.

Lower Bound

Theorem (Elkin 2006, DGPV'08)

Assuming the girth conjecture, an algorithm that computes a connected subgraph with $o(n^{1+1/k})$ edges has expected time at least k.

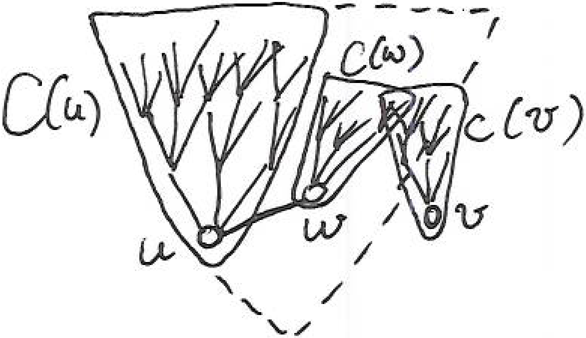
- A graph with girth greater than $2k$ looks like a tree after $t \leq k$ rounds.

Upper Bound

Theorem (DGPV'08)

It is possible to compute a $(2k-1, 0)$-spanner with $O(kn^{1+1/k})$ edges in k rounds (in $3k-1$ rounds if n is unknown).

Per Node Cluster Growth [DGPV'08]



Without Knowing n [DGPV'08]

Set $\sigma(u)$ to any value in $[\max_{v \in B(u,k-1)} |B(v,k)|^{1/k}, n^{1/k}]$.
$C(u) := \{u\}$ /* cluster around u */
$H(u) := (\{u\}, \emptyset)$ /* spanner edges selected by u */
For $i := 1$ *to* k **do**
  Node u sends $C(u)$ to all nodes in $N(u)$,
  and receives $C(v)$ from all $v \in N(u)$.
  $W(u) := N(u) \setminus \left\{ v \mid C(u) \cap C(v) \neq \emptyset \right\}$ /* nodes to cover */
  $j := 0$
  While $\exists w \in W(u)$ *and* $j < \sigma(u)$ **do**
    Pick $w \in W(u)$.
    Add edge uw to $H(u)$.
    Add $C(w)$ to $C(u)$.
    $W(u) := W(u) \setminus \left\{ v \in W(u) \mid C(v) \cap C(w) \neq \emptyset \right\}$
    $j := j + 1$

Without Knowing n (Performances)

- Time : $3k - 1$ rounds (k if n is known).
- Edges : $m(H) \le \sum_u k\sigma(u) \le kn\Delta_k^{1/k} \le kn^{1+1/k}$ where $\Delta_k = \max_u |B(u,k)|$.
- After iteration i, radius$(C(u)) \le i$.
- After iteration i, $|C(u)| \ge \max_{v \in B(u,k-i)} |B(v,k)|^{i/k}$ or $W(u) = \emptyset$.
- Stretch : at the end, $W(u) = \emptyset$ and for $v \in N(u)$, $d_H(u,v) \le 1 + (k-1) + (k-1)$.

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Related Upper Bounds

- [Baswana et al. 2007] provide an algorithm for computing a $(k, k-1)$-spanner (unweighted) with $O(kn^{1+1/k})$ expected size in $O(k)$ rounds using randomized sampling :
- at round i, a cluster is considered for growing with probability $n^{-1/k}$.
- It is randomized and requires knowledge of n.
- [Baswana et al. 2007] provide an algorithm for computing a $(2k-1, 0)$-spanner (weighted) with $O(kn^{1+1/k})$ expected size in $O(k^2)$ rounds using also randomized sampling and knowledge of n.

Toward Additive Spanners

The Open Problem

- Do $(1, 2k-2)$-spanners with $O(n^{1+1/k})$ edges exist?
- $k = 2$: yes [Aingworth & al. 1999].
- $k = 3$: ok with stretch $(1, 6)$ [Baswana & al. 2005].
- $k > 3$: ??
- Do $(1, O(1))$-spanners with $o(n^{4/3})$ edges exist?

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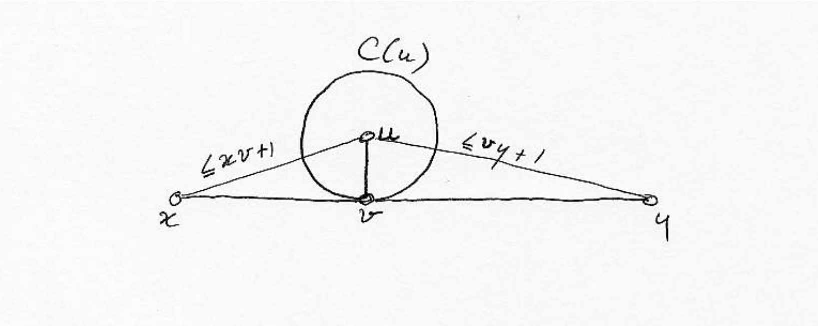
(1, 2)-spanners of size $O(n^{3/2})$ [Aingworth & al. 1999]

$G' := G$
$H := \emptyset$
While $\exists u \in V(G') \mid \deg_{G'}(u) > n^{1/2}$ **do**
  $C(u) := B_{G'}(u, 1)$ /* cluster around u */
  $G' := G' - C(u)$
  $H := H \cup BFS_G(u)$
$H := H \cup E(G')$ /* add remaining edges */

- H is a (1, 2)-spanner of G.
- $m(H) \leq 2n^{3/2}$ (at most $n^{1/2}$ clusters).

Proof of stretch



$(1,6)$-spanners of size $O(n^{4/3})$ [Baswana & al. 2005]

- Greedily compute clusters of size greater than $n^{1/3}$.
- At most $n^{2/3}$ clusters.
- Add shortest paths for reducing distances between all cluster pairs.
- This works because $\left(n^{2/3}\right)^2 = n^{4/3}$.
- For $k = 4$: $\left(n^{3/4}\right)^2 \gg n^{5/4}$...

Nearly Additive Spanners

- $(1+\varepsilon, \beta)$-spanners with $O(\beta n^{1+1/k})$ edges with $\beta = k^{\log\log k - \log\varepsilon}$ [Elkin & Peleg 2004].
- $(1+\varepsilon, O(1/\varepsilon)^{k-2})$-spanners with $O(kn^{1+1/k})$ edges [Thorup & Zwick 2006], indeed f-spanners with $f(d) = d + O\left(kd^{1-\frac{1}{k-1}}\right)$.

Distributed Nearly Additive Spanners [DGPV'09]

Set $\sigma(u)$ to a value in
$[\max_{v \in B(u,\rho[2,k])} |B(v,\rho[1,k])|^{1/k}, n^{1/k}]$
$C(u) := \{u\}$ /* cluster around u */
$F(u) :=$ FALSE /* termination flag */
$H(u) := (\{u\}, \emptyset)$ /* spanner edges selected by u */
For $i := 1$ *to* k **do**
  Node u sends $C(u), F(u)$ to all nodes in $B(u,\rho_i)$,
  and receives $C(v), F(v)$ from all $v \in B(u,\rho_i)$.
  $W(u) :=$
  $B(u,\rho_i) \setminus \{v \mid F(v) = \text{TRUE} \text{ or } C(u) \cap C(v) \neq \emptyset\}$
  $j := 0$
  While $\exists w \in W(u)$ *and* $j < \sigma(u)$ **do**
    Pick $w \in W(u)$ such that $d_G(u,w)$ is minimal.
    Add a shortest path in G from u to w to $H(u)$.
    Add $C(w)$ to $C(u)$.
    $W(u) := W(u) \setminus \{v \in W(u) \mid C(v) \cap C(w) \neq \emptyset\}$
    $j := j + 1$
  If $W(u) = \emptyset$ **then** $F(u) :=$ TRUE **else** $F(u) :=$ FALSE

Performances

- Time : $O(\rho_1 + \cdots + \rho_k)$.
- Edges : $m(H) \le (\rho_1 + \cdots + \rho_k)n^{1+1/k}$.
- After iteration i, radius$(C(u)) \le \rho_1 + \cdots + \rho_i$.

stretch	size	time	parameters
$(2k-1, 0)$	$k \cdot n^{1+1/k}$	$O(k)$	$\rho_1 = \cdots = \rho_k = 1$
$(1+\varepsilon, 2-\varepsilon)$	$(1+\lceil\frac{2}{\varepsilon}\rceil)\cdot n^{3/2}$	$O(\varepsilon^{-1})$	$\rho_1 = 1, \rho_2 = \lceil\frac{2}{\varepsilon}\rceil$, $\varepsilon \in (0,2]$
$(1+\varepsilon, 4(1+\lceil\frac{4}{\varepsilon}\rceil)^{k-2} - \varepsilon)$	$(1+\lceil\frac{4}{\varepsilon}\rceil)^{k-1} \cdot n^{1+1/k}$	$O((1+\lceil\frac{4}{\varepsilon}\rceil)^{k-1})$	$\rho_1 = 1$, $\rho_i = \lceil\frac{4}{\varepsilon}\rceil(1+\lceil\frac{4}{\varepsilon}\rceil)^{i-2}$ $\varepsilon \in (0,4]$
$(5, 2^k - 4)$	$2^{k-1} \cdot n^{1+1/k}$	$O(2^k)$	$\hookrightarrow$ with $\varepsilon = 4$
$(3, 4 \cdot 3^{k-2} - 2)$	$3^{k-1} \cdot n^{1+1/k}$	$O(3^k)$	$\hookrightarrow$ with $\varepsilon = 2$
$(2, 4 \cdot 5^{k-2} - 2)$	$5^{k-1} \cdot n^{1+1/k}$	$O(5^k)$	$\hookrightarrow$ with $\varepsilon = 1$

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$(2, 4 \cdot 5^{k-2} - 2)$	$5^{k-1} \cdot n^{1+1/k}$	$O(5^k)$	$\hookrightarrow$ with $\varepsilon = 1$

Conclusion

- A challenge : dynamic distributed compact routing.
- A first step : distributed spanner construction.
- An open problem : existence of sparse additive spanners.

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Spanner Variants

Spanners and link state routing

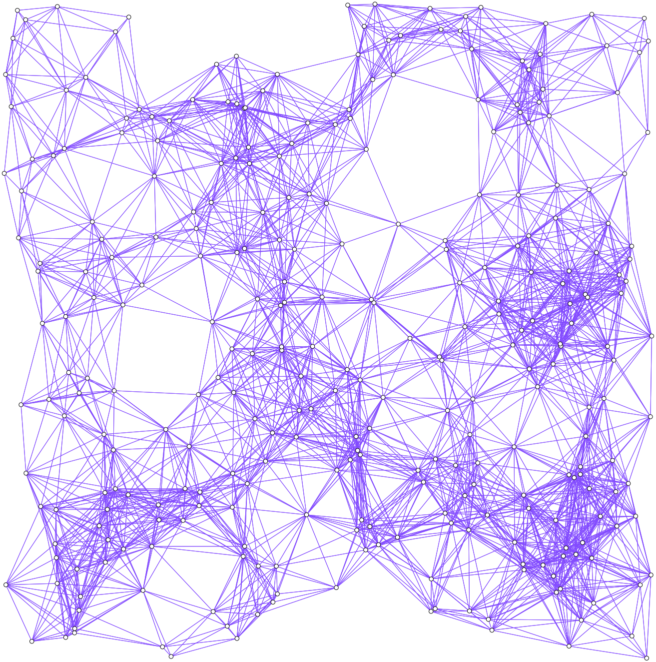
Link state routing :

- Each node discovers its neighbors,
- and advertises the state of some neighboring links.

Optimize link state advertisements :

- few links (flooded information),
- efficient routes.

Dense network G



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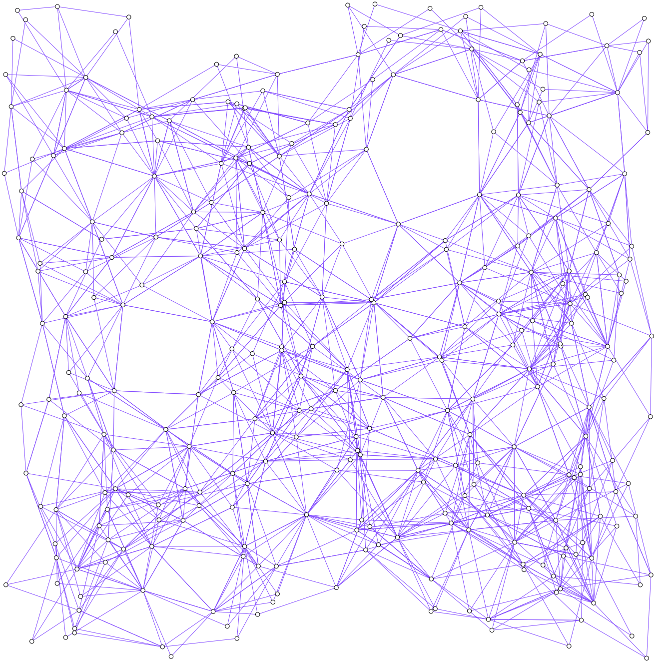
Sub-Graph H



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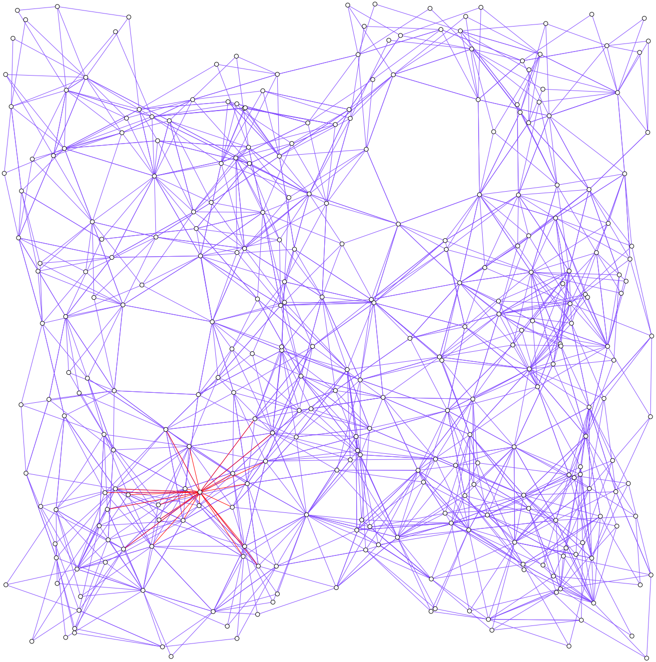
Augmented sub-graph H_u



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Remote Spanners

Definition (Remote Spanner, JV'09)

$H \subseteq G$ is an (α,β)-remote-spanner of G iff $d_{H_u}(u,v) \leq \alpha \cdot d_G(u,v) + \beta$ for all u,v where $H_u = H \cup \{uv \mid v \in N(u)\}$.

OLSR relies on the construction of a $(1,0)$-remote-spanner.

Remote Spanners

Theorem (JV'09)

- *An (α, β)-spanner is an $(\alpha, \beta - \alpha + 1)$-remote-spanner implying the existence of $(k, 0)$-spanner with $O(kn^{1+1/k})$ edges using [Baswana & al. 2005].*
- *A random unit disk graph has a $(1, 0)$-remote-spanner with $O(n^{4/3})$ edges in expectation.*
- *A $(1, 0)$-remote-spanner with size $O(\log n)$ from optimal can distributively be computed in $O(1)$ time.*

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Remote Spanners

Theorem (JV'09)

- *If G is the unit ball graph of a doubling metric with dimension p (distances are unknown), a $(1+\varepsilon, 1-2\varepsilon)$-remote-spanner with $O(n\varepsilon^{-(p+1)})$ edges can be computed in $O(\varepsilon^{-1})$ time.*
- *If G is the unit ball of a doubling metric, a 2-multipath $(2,-1)$-remote-spanner ($\varepsilon = 1$) with $O(n)$ edges can be computed in $O(1)$ time.*
- *A c-multipath $(1,0)$-remote-spanner with size $O(\log n)$ from optimal can distributively be computed in $O(1)$ time.*

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Spanner of Directed Graphs

Definition (Roundtrip Distance, Cowen & Wagner 1999)

In a strongly connected graph G the roundtrip distance $d_G(u,v)$ is the weight of a lightest circuit traversing u and v :

$$d_G(u,v) = \overrightarrow{d_G}(u,v) + \overrightarrow{d_G}(v,u)$$

Theorem (Roditty & al. 2002)

Every graph has a $(3,0)$-roundtrip-spanner with $O(n^{3/2})$ edges and a $(2k+\varepsilon,0)$-roundtrip-spanner with $O(\frac{k^2}{\varepsilon}n^{1+1/k}\log nW)$ edges where the weights are in the range $[1,W]$.

Spanner of Directed Graphs

Definition (Roundtrip Distance, Cowen & Wagner 1999)

In a strongly connected graph G the roundtrip distance $d_G(u,v)$ is the weight of a lightest circuit traversing u and v :

$$d_G(u,v) = \overrightarrow{d_G}(u,v) + \overrightarrow{d_G}(v,u)$$

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Multipath Spanners

Definition (Multipath Distance, JV'09)

The c-multipath distance $d^c_G(u, v)$ is the weight of the lightest collection of c disjoint paths from u to v.

Definition (Multipath Spanner, JV'09)

$H \subseteq G$ is a c-multipath (α, β)-spanner of G iff $d^i_H(u, v) \leq \alpha \cdot d^i_G(u, v) + i\beta$ for all u, v and $i \leq c$.

Theorem (GGV'10)

Every graph has a 2-multipath $(3, 0)$-spanner with $O(n^{3/2})$ edges and a c-multipath $(c(2k-1), 0)$-spanner with $O(cn^{1+1/k})$ edges (edge disjoint paths are considered).

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Fault Tolerant Spanner

Definition (Fault Tolerant Spanner, Chechik & al. 2009)

$H \subseteq G$ is a f-fault tolerant (α, β)-spanner of G iff $d_{H-F}(u,v) \leq \alpha d_{G-F}(u,v) + \beta$ for all u, v and $F \subset V(G)$ with $n(F) \leq f$.

Theorem (Chechik & al. 2009)

Every graph has an f-fault tolerant $(2k-1, 0)$-spanner with $O(f^3 k^{f+1} \cdot n^{1+1/k} \log^{1-1/k} n)$ edges.

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Distance Emulators

Definition (Distance Eumlator, Dor & al. 2000)

H is an (α, β)-emulator of G iff for all u, v,
$d_G(u, v) \leq d_H(u, v) \leq \alpha \cdot d_G(u, v) + \beta$.

Theorem (Dor & al. 2000)

Every graph has $(1, 4)$-emulator with $O(n^{3/2})$ edges.

Theorem (Thorup & Zwick 2006)

Every graph has an f-emulator with $O(kn^{1+\frac{1}{2^k-1}})$ edges where $f(d) = d + O\left(kd^{1-\frac{1}{k-1}}\right)$.

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Distance Preservers

Definition (Bollobás & al. 2003)

$H \subseteq G$ is a D-preserver iff $d_H(u,v) = d_G(u,v)$ for all u, v such that $d_G(u,v) \geq D$.

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Conclusion

There are still new spanner algorithms to find.