Diffusion and Cascading Behavior in Random Networks

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Séminaire Complex networks LIP6.

Crossing the Chasm

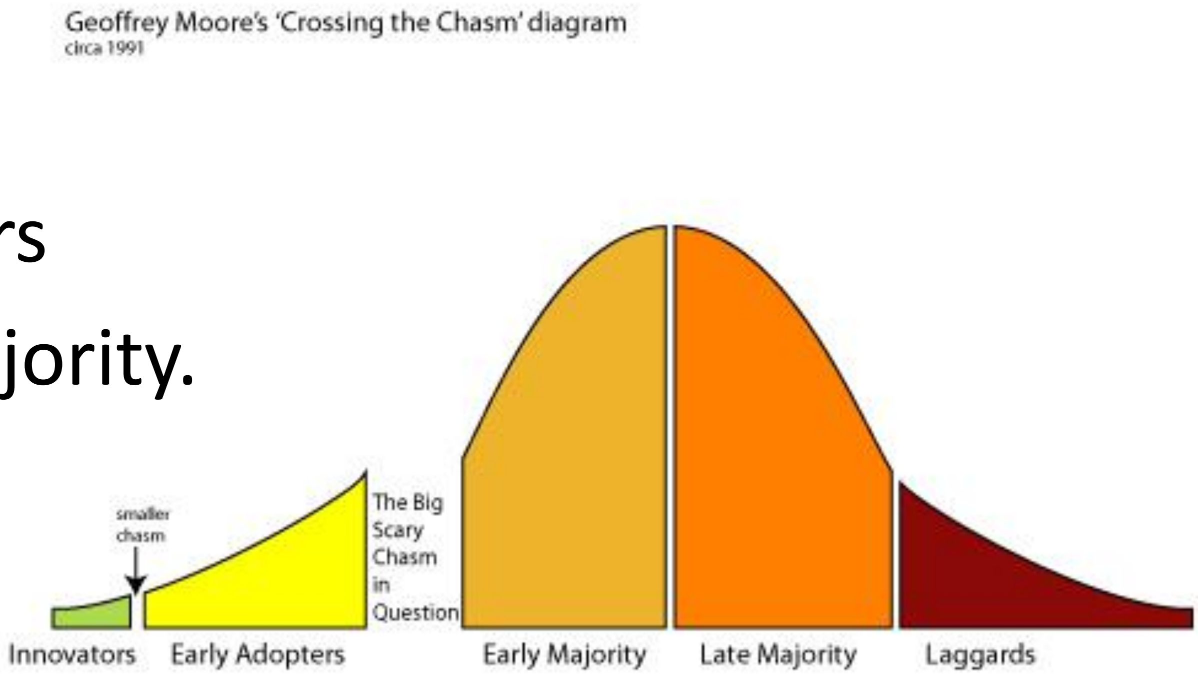
- Diffusion of innovations theory of Everett Roger.
- There is a

chasm between

the early adopters

and the early majority.

(Moore 2002)



Explaining the chasm

- Standard epidemic models (SIR/S) do not capture the chasm…
- … at least on uniform random graphs.

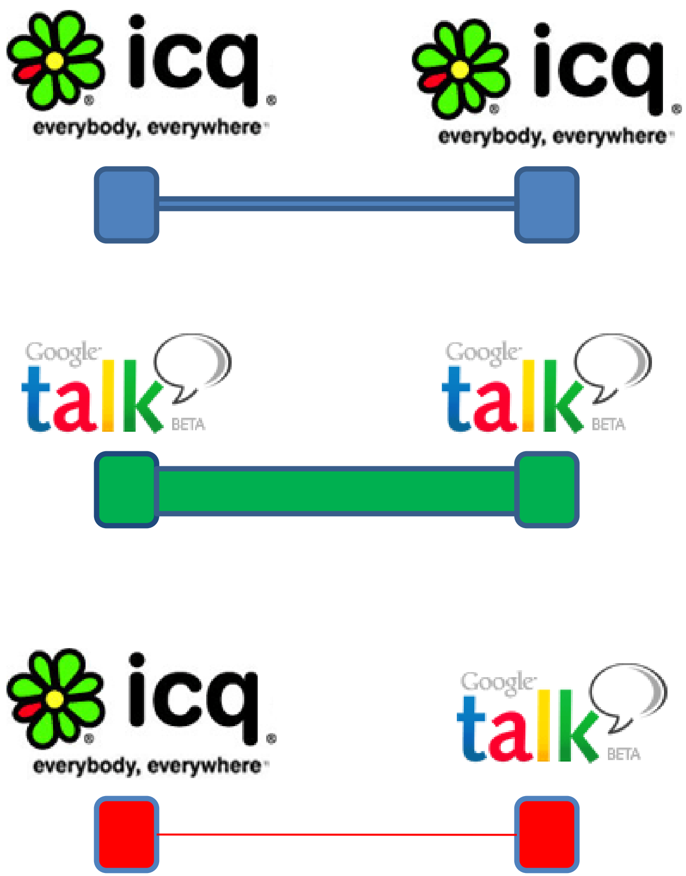
  chasm due to clustering?

- Different idea: add neighborhood effects.

Disposition towards adoption:

$$f_d(r) = \alpha r \quad \Longrightarrow \quad f_d(r) = \begin{cases} 0 & \text{if } r/d < \theta \\ 1 & \text{if } r/d \geq \theta \end{cases}$$

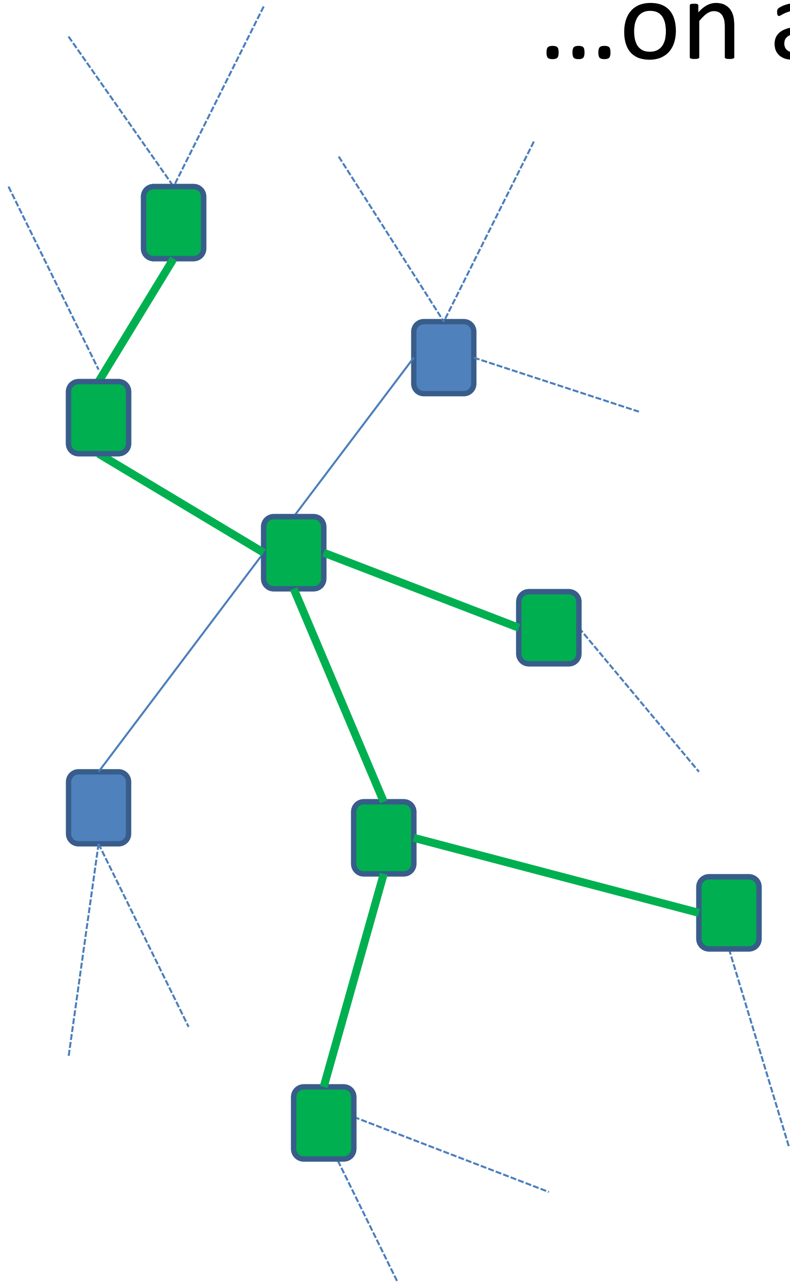
Game-theoretic diffusion model…



- Both receive payoff q.
- Both receive payoff 1-q>q.
- Both receive nothing.

Morris (2000)

…on a network.



- Everybody start with icq everybody, everywhere
- Total payoff = sum of the payoffs with each neighbor.
- A fraction of the population is forced to Google talk BETA
- If 2(1-q)>3q, i.e.

  2>5q

Threshold Model

- State of agent i is represented by

$$X_i = \begin{cases} 0 & \text{if icq} \\ 1 & \text{if Google talk} \end{cases}$$

- Switch from icq (everybody, everywhere) to Google talk (BETA) if:

$$\sum_{j \sim i} X_j \geq \theta(d_i)$$

- In previous case: $\theta(d) = dq$

(1) Model

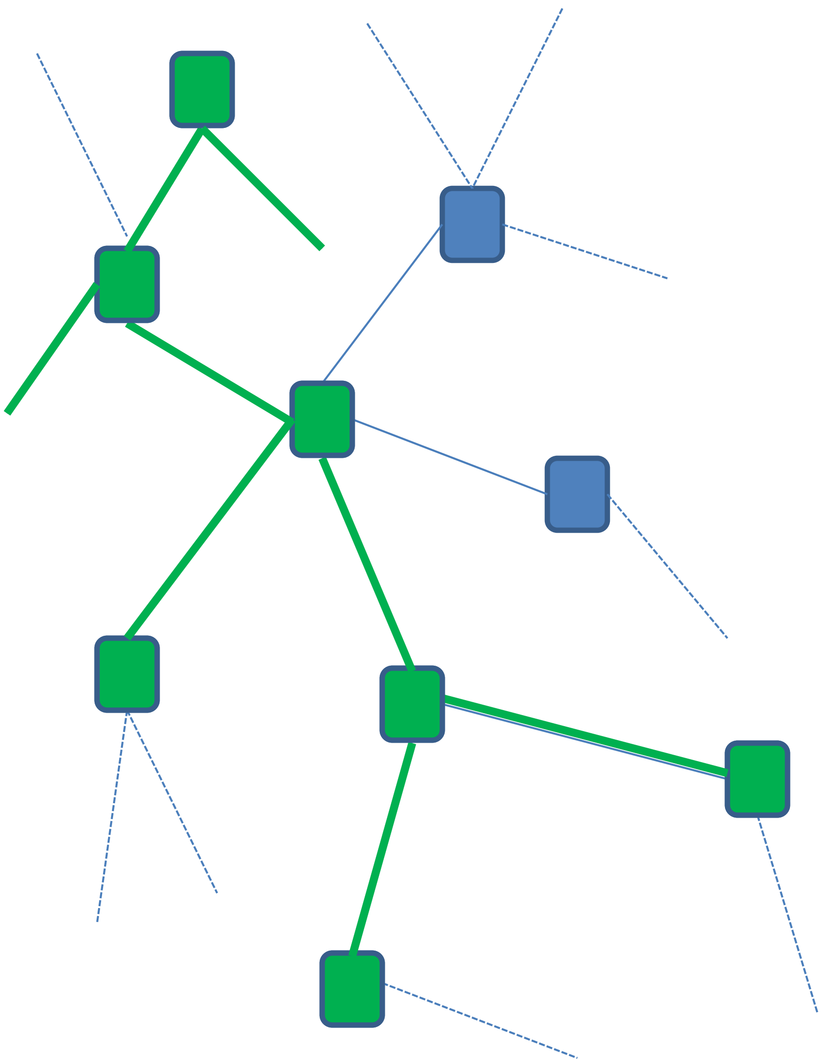
(2) Results

(3) Proofs

(1) Random networks.

- Random graphs with given degree sequence introduced by Molloy and Reed (1995).
- In this talk:
  - Random regular graphs. $d_i = d$
  - Erdös-Réyni graphs, G(n,p/n). $d_i \approx Poi(p)$
- We are interested in large population asymptotics. n= number of vertices tends to infinity.

(1) Percolated Threshold Model



- Bond percolation with proba. $1-\pi$
- Symmetric threshold epidemic:

$$\sum_{j \sim i} X_j \geq K_i(d_i)$$

- Seed of active nodes

(1) Versatile model for epidemics

- Null threshold = contact process
- No bond percolation = bootstrap percolation
- Some easy general results:
  - Monotonicity: only transition passive to active.
  - In a finite graph, there is only one possible final state for the epidemic.
- I will concentrate on properties of the final state, for large random graphs.

(1) Model

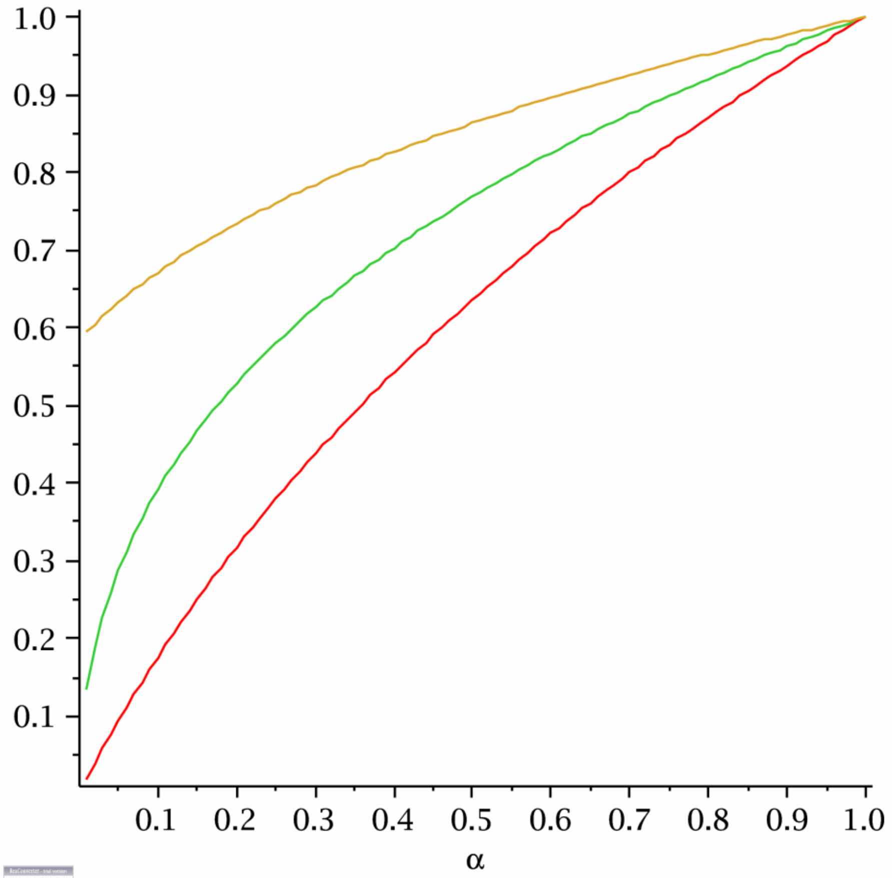
(2) Results

(3) Proofs

(2) Sanity check!

- Take $\theta(d) = 0$ and we obtain a simple exploration of the connected components of the graph.

E-R model with
p=0.5;
p=1;
p=1,5.



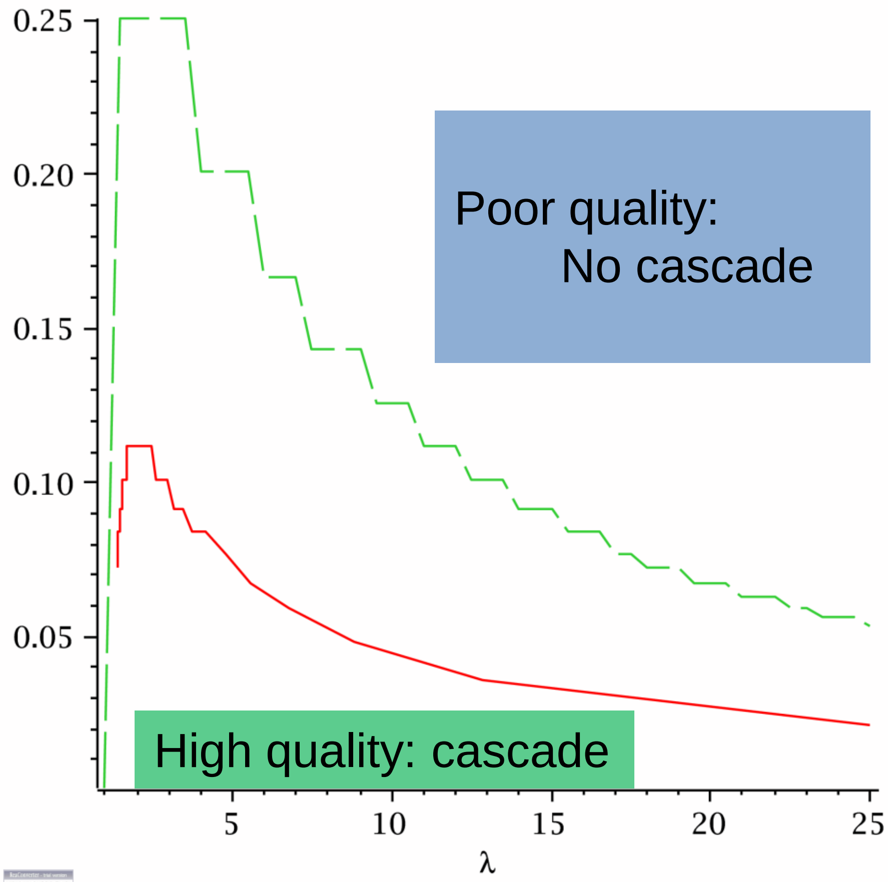
In accordance with Molloy and Reed!

(2) Cascade capacity

- Definition: maximum q for which a single individual can trigger a global cascade.
- Low q = high quality

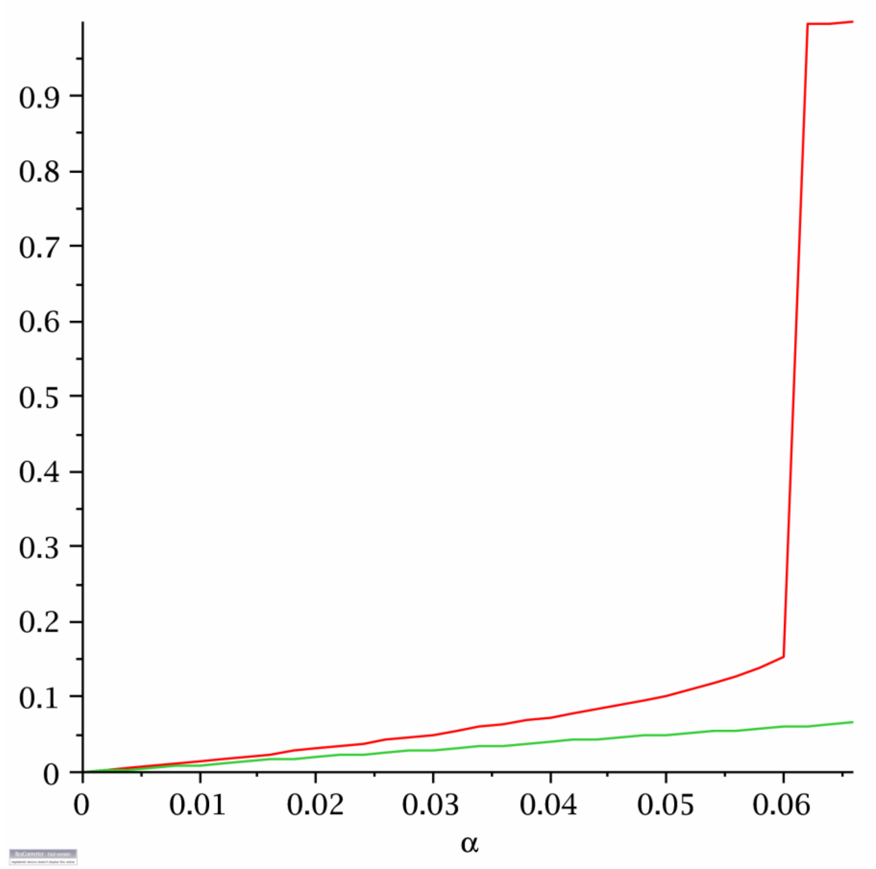
$$\theta(d) = dq$$

In accordance with Watts (2002)



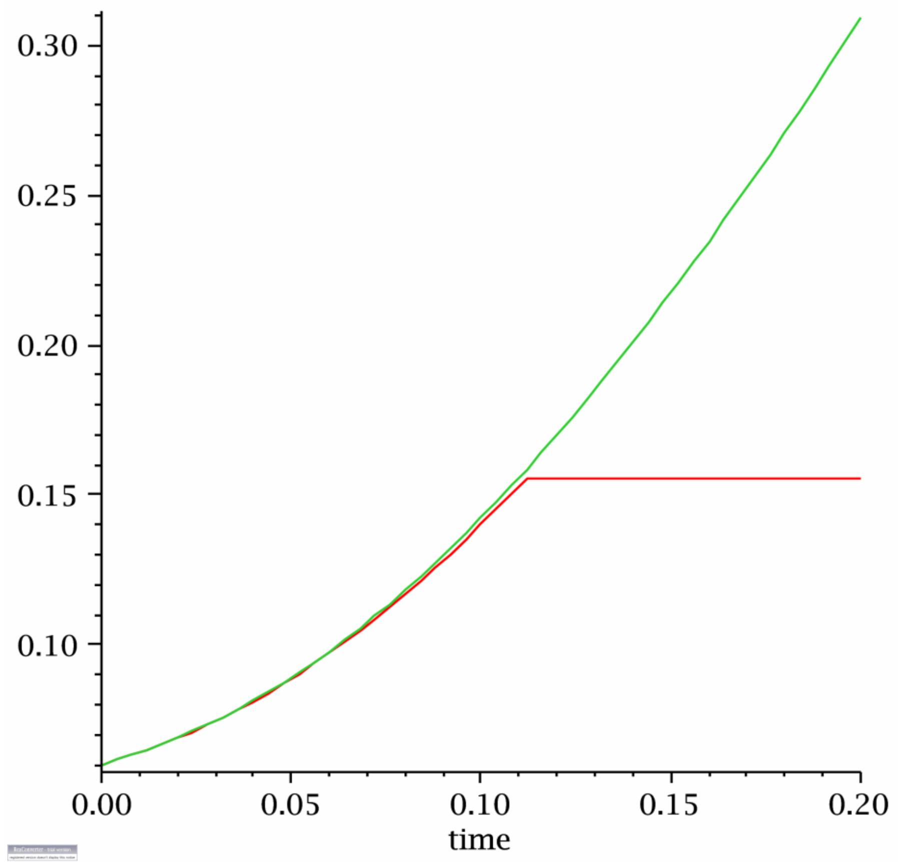
(2) Phase transition

- What happens when q is bigger than the cascade capacity?



(2) Back to the chasm

- α is the proportion of early adopters
- q quantifies the impact of marketing



There is a threshold phenomenon:
- for q>s, the early adopters do not contaminate the majority.
-for q<s, they do!

s quantifies the size of the chasm.

(2') Vaccination

(2’) Vaccination and Attack

- Perfect vaccine: remove vaccinated population from the graph (site percolation).
- Acquaintance vaccination: Sample each node uniformly and inoculate a neighbor of this node taken at random.
- Degree based attack: randomly attack a node with a probability depending on its degree.

(2’) Cascade Condition

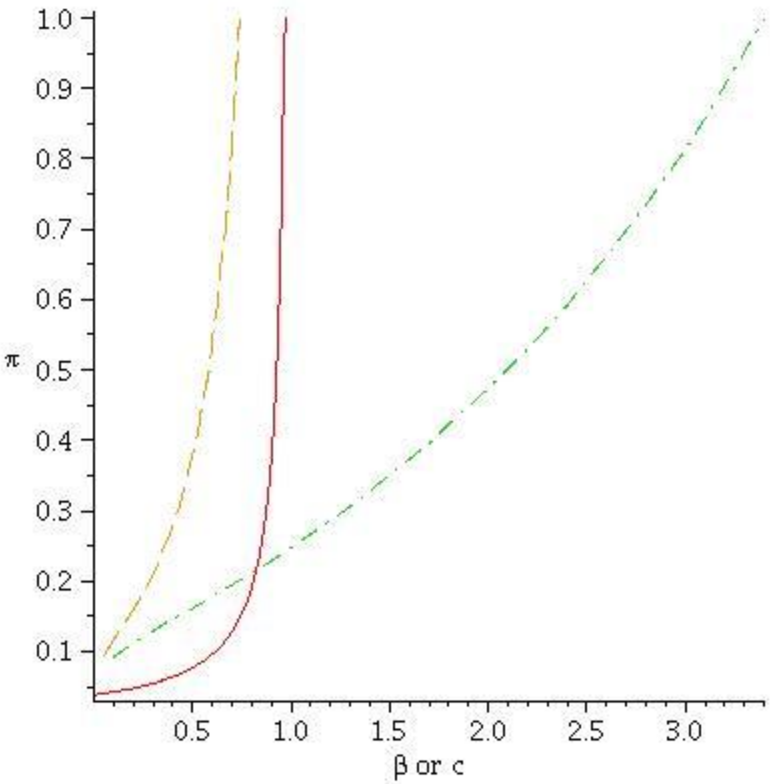
- Random graph with degree distribution: D

  (configuration model: Molloy-Reed 95)
- Bond percolation: π and threshold: $K(d)$.
- When can a single active node have a global impact?

$$\pi \mathbb{E}[D(D-1)\mathbb{1}(K(D)=0)] > \mathbb{E}[D]$$

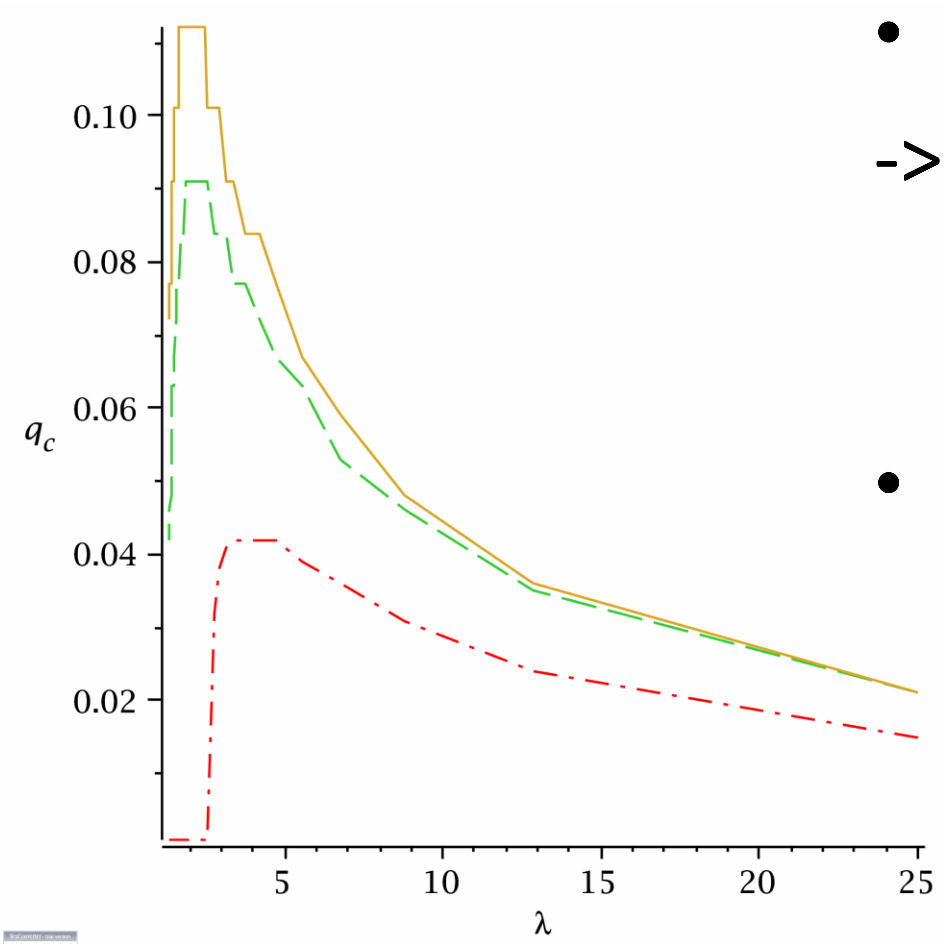
- $K \equiv 0$ Epidemic contagion threshold.

(2’) Vaccination for the contact process



- Epidemic threshold as a function of vaccinated population.
- If $\mathbb{E}[D^2] = \infty$, uniform vaccination is useless. Acquaintance vaccination can stop epidemic!

(2') Vaccination for threshold model



- Threshold $K(d) = qd$

-> become active when fraction of active neighbors $\geq q$

- Contagion threshold as a function of mean degree.

(1) Model

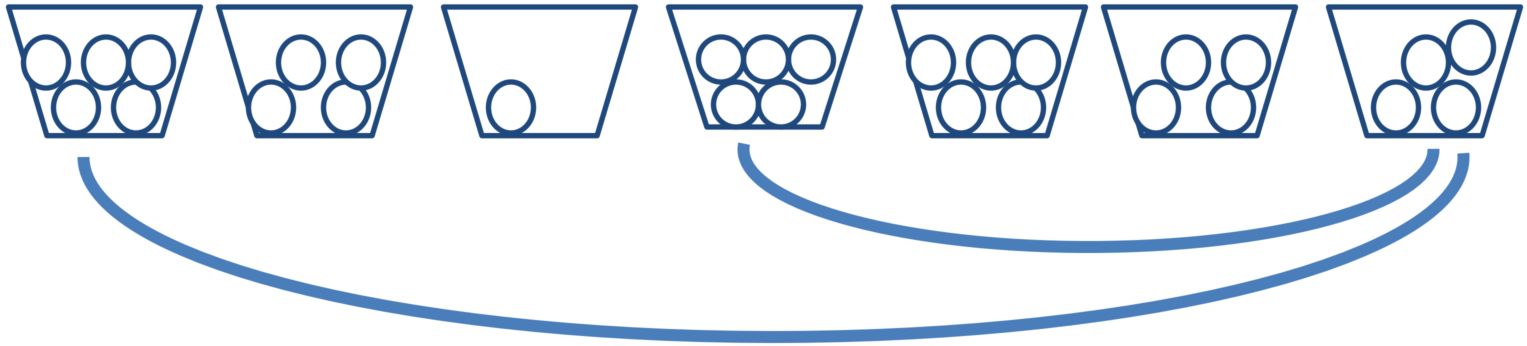
(2) Results

(3) Proofs

Configuration Model

- Vertices = bins and half-edges = balls

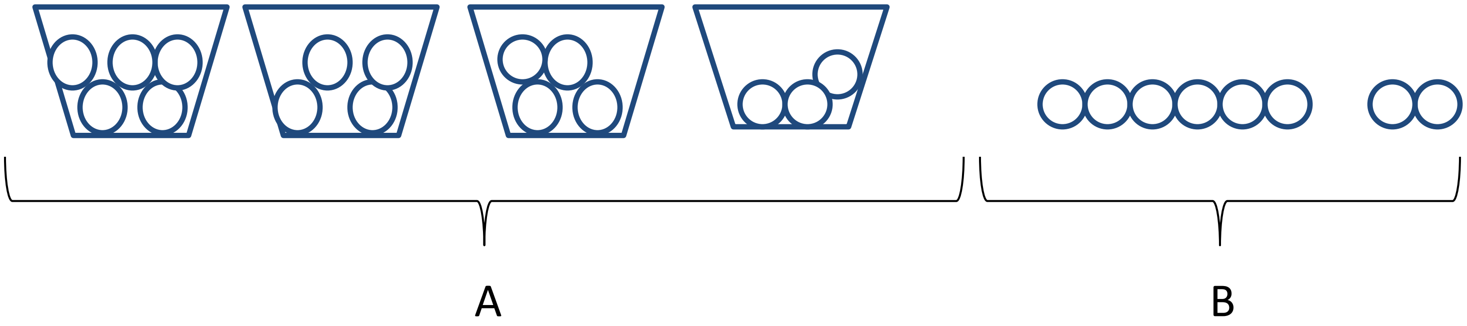
Bollobás (80)



Coupling

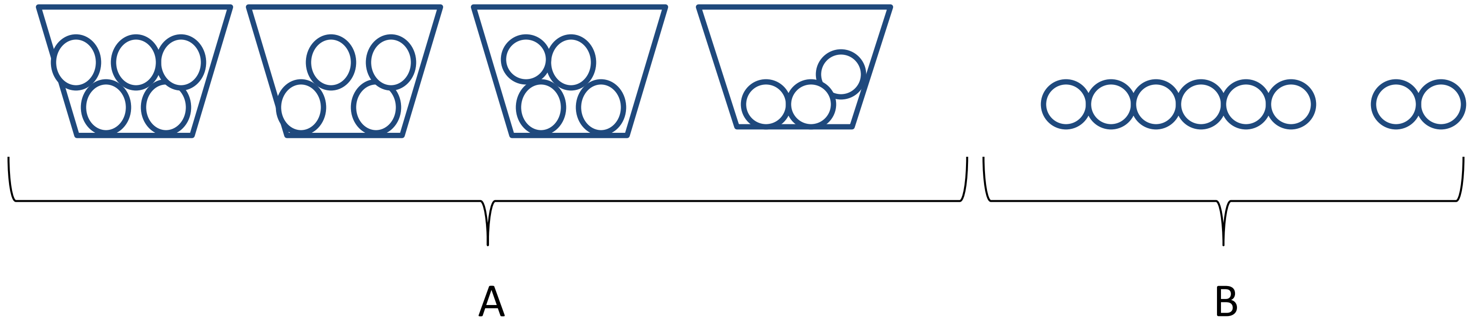
- Type A if $d_i^A \leq d_i - K_i$

Janson-Luczak (07)



Deletion in continuous time

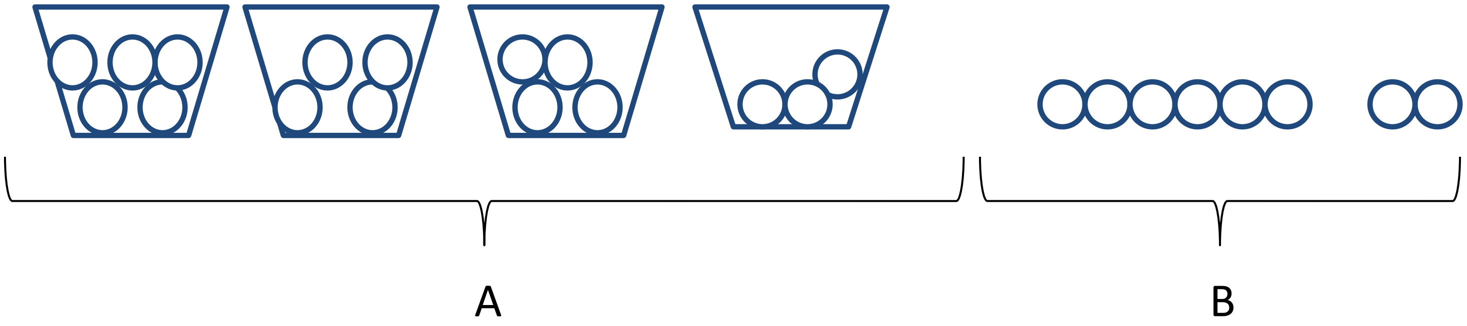
- Each white ball has an exponential life time.



Percolated threshold model

- Bond percolation: immortal balls

A



Death processes

- Rate 1 death process (Glivenko-Cantelli):

$$\sup_{t \geq 0} |N^{(n)}(t)/n - e^{-t}| \xrightarrow{p} 0$$

- Death process with immortal balls:

$$\frac{U_{s\ell,r}(t)}{n} \sim p_{s\ell} b_{sr} (1 - \pi + \pi e^{-t})$$

Death Processes for white balls

- For the white A and B balls:

$$\frac{A(t)+B(t)}{n} \sim \lambda e^{-t}(1-\pi+\pi e^{-t}).$$

- For the white A balls:

$$\frac{A(t)}{n} \sim \sum_{s,r \geq s-\ell} r(1-\alpha_s)p_{s\ell}b_{sr}(1-\pi+\pi e^{-t}).$$

Epidemic Spread

- $\widehat{z}$ largest solution in [0,1] of:

$$\lambda z(1-\pi+\pi z)-\sum_{s,r\geq s-\ell} r(1-\alpha_s)p_{s\ell}b_{sr}(1-\pi+\pi z)=0.$$

$$h_1(z)=\sum_{s,r\geq s-\ell}(1-\alpha_s)p_{s\ell}b_{sr}(1-\pi+\pi z).$$

- If $\widehat{z}=0$, then final outbreak: $1-h_1(0)$
- If $\widehat{z}=0$, and not local minimum, outbreak:
$1-h_1(\widehat{z})$

(3) Branching Process Approximation

- Local structure of G = random tree
- Recursive Distributional Equation:

$$Y_i = 1 - (1 - \sigma_i) \mathbb{1} \left(\sum_{\ell \to i} B_{\ell i} Y_\ell \leq K(d_i) \right)$$

(3) Solving the RDE

$$z = \mathbb{P}(Y = 0)$$

$$\lambda z(1 - \pi + \pi z) = h(z)$$

$$h(z) = \sum_{s, r \geq s - \ell} r(1 - \alpha_s) p_{s\ell} b_{sr} (1 - \pi + \pi z)$$

Conclusion

- The locally tree-like structure gives the intuition for the solution…
- … but not the proof!
- Configuration model + results of Janson and Luczak.
- Generic epidemic model which allows to retrieve basic results for random graphs…
- …. and new ones: contagion threshold, phase transitions.